MATHEMATICS

SECTION- I

STRAIGHT OBJECTIVE TYPE

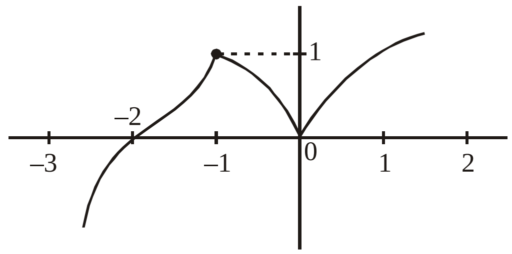
This section contains 6 multiple choice questions numbered 1 to 6. Each question has 4 choice (A), (B), (C) and (D), out of which ONLY-ONE is correct

1. The total number of local maxima and local minima of the function

$$f(x) = \begin{cases} (2+x)^3, & -3 < x \le -1 \\ x^{2/3}, & -1 < x < 2 \end{cases} \text{ is}$$

(A) 0 (B) 1 (C) 2 (D) 3

Sol: Ans [C]



So total number of local maximum and local minima = 2

2. Let a and b be non-zero real numbers. Then, the equation

$(ax^2 + by^2 + c)(x^2 - 5xy + 6y^2) = 0$

represents

(A) four straight lines, when $c = 0$ and a, b are of the same sign

(B) two straight lines and a circle, when $a = b$, and c is of sign opposite to that of a

(C) two straight lines and a hyperbola, when a and b are of the same sign and c is of sign opposite to that of a

(D) a circle and an ellipse, when a and b are of the same sign and c is the sign opposite to that of a

Sol: Ans [B]

If $a = b$ is c is of opposite sign that of a then $ax^2 + by^2 + c = 0$ will represent a circle.

And $x^2 - 5xy + 6y^2 = 0$

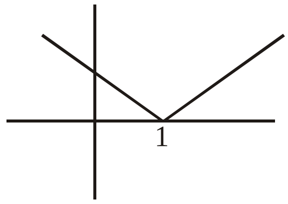
$\Rightarrow$ $(x - 2y)(x - 3y) = 0$ will represent two straight line.

3. Let $g(x) = \dfrac{(x-1)^n}{\log \cos^m (x-1)}$; $0 < x < 2$, m and n are integers, $m \ne 0$, $n > 0$ and let p be the left hand derivative of $|x - 1|$ at $x = 1$. If $\lim\limits_{x \to 1^+} g(x) = p$, then

(A) $n = 1, m = 1$ (B) $n = 1, m = -1$ (C) $n = 2, m = 2$ (D) $n > 2, m = n$

Sol: Ans [C]

$$f(x) = |x-1| = \begin{cases} x-1, & x \geq 1 \\ -x+1, & x < 1 \end{cases}$$



$$P = \text{L.H.D.} = \lim_{x \to 1^-} f'(x)$$

$$= \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{-(1-h) + 1 - 0}{-h}$$

$$= \lim_{h \to 0} \frac{-1 + h + 1}{-h} = -1$$

$$\lim_{x \to 1^+} g(x) = \lim_{h \to 0} g(1+h)$$

$$= \frac{(1+h-1)^n}{\log \cos^m (1+h-1)}$$

$$= \lim_{h \to 0} \frac{h^n}{\log \cos^m (h)}$$

$$= \lim_{h \to 0} \frac{nh^{n-1}}{\frac{1}{\cos^m h} \cdot m \cos^{m-1} n(-\sin h)}$$

$$= \lim_{h \to 0} \frac{-nh^{n-1} \cos h}{m \sin h}$$

If $n = 2$, $m = 2$ then, $\lim_{x \to 1^+} g(x) = -1$

4. If $0 < x < 1$, then

$$\sqrt{1+x^2}\,[\{x\cos(\cot^{-1}x) + \sin(\cot^{-1}x)\}^2 - 1]^{1/2} =$$

(A) $\dfrac{x}{\sqrt{1+x^2}}$ (B) x (C) $x\sqrt{1+x^2}$ (D) $\sqrt{1+x^2}$

Sol: Ans [C]

$\cot^{-1} x = \theta$

$x = \cot\theta$

$$\sqrt{1+\cot^2\theta}\,[(\cot\theta\cos\theta + \sin\theta)^2 - 1]^{1/2} = \operatorname{cosec}\theta \left[\frac{1-\sin^2\theta}{\sin^2\theta}\right]^{1/2}$$

$$= \cot\theta \operatorname{cosec}\theta$$

$$= x\sqrt{1+x^2}$$

5. The edges of a parallelopiped are of unit length and are parallel to non-coplanar unit vector $\hat{a}, \hat{b}, \hat{c}$ such that

$$\hat{a} \cdot \hat{b} = \hat{b} \cdot \hat{c} = \hat{c} \cdot \hat{a} = \frac{1}{2}.$$

Then, the volume of the parallelopiped is

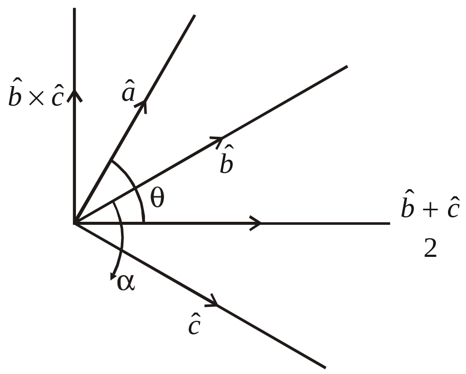
(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2\sqrt{2}}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{1}{\sqrt{3}}$

Sol: Ans [A]

$\hat{b} \cdot \hat{c} = \frac{1}{2}$ (Let α is the angle between $\hat{b}$ and $\hat{c}$)

$\Rightarrow 1 \cdot 1 \cos\alpha = \frac{1}{2}$

$\Rightarrow \alpha = \pi/3$



$\hat{a} \cdot \frac{(\hat{b}+\hat{c})}{2} = \frac{\hat{a}\cdot\hat{b} + \hat{a}\cdot\hat{c}}{2} = \frac{1}{2}$

$1 \cdot \frac{\sqrt{3}}{2} \cos\theta = \frac{1}{2}$

$\cos\theta = \frac{1}{\sqrt{3}}$

$[\hat{a}\ \hat{b}\ \hat{c}] = |\ \hat{a} \cdot (\hat{b} \times \hat{c})\ | = |\ \hat{a}\ |\,|\ \hat{b} \times \hat{c}\ | \cos(90 - \theta)$

$= 1\ |\ \hat{b} \times \hat{c}\ | \sin\theta$

$\hat{b} \times \hat{c} = 1 \cdot 1 \cdot \sin 60\ \hat{n} = \frac{\sqrt{3}}{2} \hat{n}$

$|\hat{b} \times \hat{c}| = \frac{\sqrt{3}}{2}$

$\sin\theta = \sqrt{1 - \frac{1}{3}} = \frac{\sqrt{2}}{\sqrt{3}}$

$\Rightarrow [\hat{a}\ \hat{b}\ \hat{c}] = \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{\sqrt{3}} = \frac{1}{\sqrt{2}}$

6. Consider the two curves

$C_1 : y^2 = 4x$

$C_2 : x^2 + y^2 - 6x + 1 = 0$

Then,

(A) C_1 and C_2 touch each other only at one point

(B) C_1 and C_2 touch each other exactly at two point

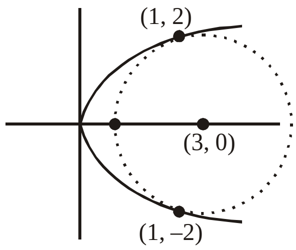
(C) C_1 and C_2 intersect (but do not touch) at exactly two points

(D) C_1 and C_2 neither intersect nor touch each other

Sol: Ans [B]

$x^2 + y^2 - 6x + 1 = 0$

$(3, 0)$, $\quad r = \sqrt{9-1} = \sqrt{8}$



$x^2 + 4x - 6x + 1 = 0$

$x^2 - 2x + 1 = 0$

$x = 1, 1$

$y = \pm 2$

$\Rightarrow$ Circle and parabola touch at (1, 2) and (1, –2).

SECTION- II

MULTIPLE CORRECT ANSWERS TYPE

This section contains 4 multiple correct answer(s) type questions. Each question has 4 choice (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

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7. Let $f(x)$ be an non-constant twice differentiable function defined on $(-\infty, \infty)$ such that $f(x) = f(1-x)$ and $f'\left(\frac{1}{4}\right) = 0$. Then,

(A) $f''(x)$ vanishes at least twice on [0, 1]

(B) $f'\left(\frac{1}{2}\right) = 0$

(C) $\displaystyle\int_{-1/2}^{1/2} f\left(x + \frac{1}{2}\right)\sin x \, dx = 0$

(D) $\displaystyle\int_{0}^{1/2} f(t)e^{\sin \pi t} \, dt = \int_{1/2}^{1} f(1-t)e^{\sin \pi t} \, dt$

Sol: Ans [A,B,C,D]

$f(x) = f(1-x)$

$\Rightarrow f'(x) = -f'(1-x)$

Putting $x = \frac{1}{2}$, $f'\left(\frac{1}{2}\right) = -f'\left(\frac{1}{2}\right) \Rightarrow \quad f'\left(\frac{1}{2}\right) = 0$

(B) is correct

Also putting $x = \frac{1}{4}$, $f'\left(\frac{1}{4}\right) = -f'\left(\frac{3}{4}\right) = 0$

$\Rightarrow f'\left(\frac{1}{4}\right) = 0 = f'\left(\frac{3}{4}\right)$

So $f'\left(\frac{1}{4}\right) = f'\left(\frac{1}{2}\right) = f'\left(\frac{3}{4}\right) = 0$

So by Rolle's Theorem, $f''(x)$ vanishes at least once in $\left[\frac{1}{4}, \frac{1}{2}\right]$ and at least once in $\left[\frac{1}{2}, \frac{3}{4}\right]$

$\Rightarrow$ $f''(x)$ vanishes at least twice in [0, 1]

(A) is correct

Now $f(x) = f(1 - x)$

Replacing x by $x + \frac{1}{2}$,

$$f\left(x + \frac{1}{2}\right) = f\left(\frac{1}{2} - x\right)$$

Now

$$I = \int_{-1/2}^{1/2} f\left(x + \frac{1}{2}\right)\sin x \; dx$$

$$= \int_{-1/2}^{1/2} f\left(\frac{1}{2} - x\right)\sin(-x) \; dx$$

(Replacing x by $-\frac{1}{2} + \frac{1}{2} - x$ i.e., $-x$ as $\int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx$)

$$= \int_{-1/2}^{1/2} f\left(\frac{1}{2} + x\right)\sin x \, dx$$

$$= -I$$

$\Rightarrow$ $I = 0$ **(C) is correct**

Now

$$I_1 = \int_0^{1/2} f(t)e^{\sin \pi t} \; dt$$

Putting $u = t + 1/2$ $\Rightarrow$ $du = dt$

$$I_1 = \int_{1/2}^{1} f\left(u - \frac{1}{2}\right)e^{\sin \pi(u - 1/2)} \; du$$

$$= \int_{1/2}^{1} f\left(u - \frac{1}{2}\right)e^{-\cos \pi u} \; du$$

$$I_2 = \int_{1/2}^{1} f(1 - t)e^{\sin \pi t} \; dt$$

$$= \int_{1/2}^{1} f\left(1-\left(\frac{3}{2}-t\right)\right) e^{\sin\pi\left(\frac{3}{2}-t\right)}\, dt$$

(Replacing t by $\frac{1}{2}+1-t$ i.e., $\frac{3}{2}-t$ as $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$)

$$= \int_{1/2}^{1} f\left(t-\frac{1}{2}\right) e^{\sin\left(\frac{3\pi}{2}-\pi t\right)}\, dt$$

$$= \int_{1/2}^{1} f\left(t-\frac{1}{2}\right) e^{-\cos\pi t}\, dt$$

$\Rightarrow \quad I_1 = I_2$ **(D) is correct**

8. Let

$$S_n = \sum_{k=1}^{n} \frac{n}{n^2+kn+k^2} \text{ and } T_n = \sum_{k=0}^{n-1} \frac{n}{n^2+kn+k^2}$$

for $n = 1, 2, 3, \ldots$. Then,

(A) $S_n < \frac{\pi}{3\sqrt{3}}$ (B) $S_n > \frac{\pi}{3\sqrt{3}}$ (C) $T_n < \frac{\pi}{3\sqrt{3}}$ (D) $T_n > \frac{\pi}{3\sqrt{3}}$

Sol: Ans [A,D]

$$S_n = \sum_{k=1}^{n} \frac{n}{n^2+kn+k^2} = \sum_{k=1}^{n} \frac{1}{n}\left(\frac{1}{1+(k/n)+(k/n)^2}\right)$$

Conisider a function $f(x) = \frac{1}{1+x+x^2}$

$$f'(x) = \frac{-(1+2x)}{(1+x+x^2)^2} < 0 \text{ for } x > 0$$

So $f(x)$ is a decreasing function.

So $S_n = \sum_{m=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right)$

Now putting $k = 1$,

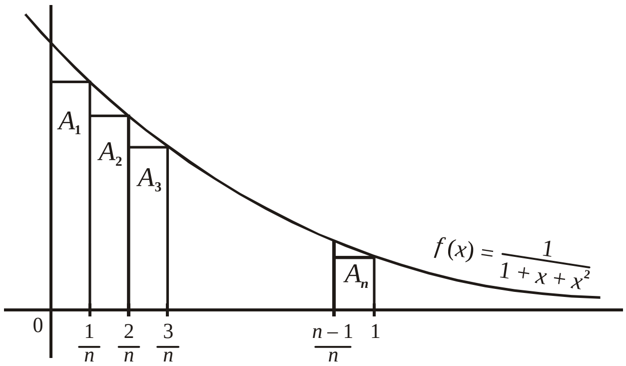
Ist term of $S_n = \frac{1}{n} f\left(\frac{1}{n}\right)$, represents area of Ist strip as shown in figure as A_1.

Putting $k = 2$,

2^{nd} term of $S_n = \frac{1}{n} f\left(\frac{2}{n}\right)$, represents area of 2^{nd} strip as shown in figure as A_2.

Similarly putting $k = n$,

last term of $S_n = \frac{1}{n} f(1)$ represents area of n^{th} strip as shown in figure as A_n.



So $A_1 + A_2 + ... + A_n < \int_0^1 \frac{1}{1+x+x^2}\,dx$

$$= \int_0^1 \frac{1}{(x+1/2)^2 + 3/4}\,dx$$

$$= \frac{1}{\sqrt{3}/2} \tan^{-1}\left(\frac{x+1/2}{\sqrt{3}/2}\right)\Bigg|_0^1 = \frac{\pi}{3\sqrt{3}}$$

So $S_n < \frac{\pi}{3\sqrt{3}}$ **(A) is correct.**

$$T_n = \sum_{k=0}^{n} \frac{n}{n^2 + kn + k^2} = \sum_{k=0}^{n} \frac{1}{n}\left(\frac{1}{1+(k/n)+(k/n)^2}\right)$$

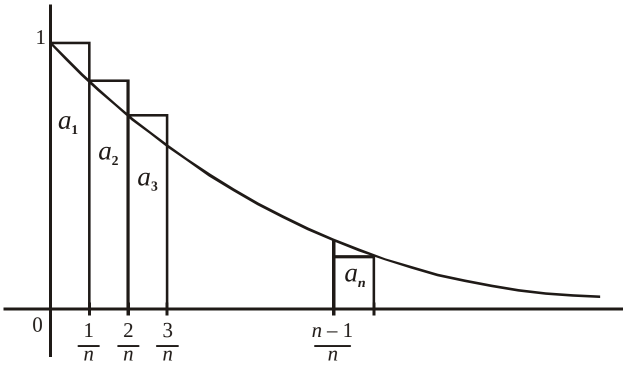
$$= \sum_{k=0}^{n-1} \frac{1}{n} f\left(\frac{k}{n}\right)$$

Putting $k = 0$, 1^{st} term of $T_n = \frac{1}{n} f(0)$ represents area of I^{st} strip as shown in graph as a_1

Putting $k = 1$, 2^{nd} term of $T_n = \frac{1}{n} f\left(\frac{1}{n}\right)$ represents area of 2^{nd} strip as shown in graph as a_2

Putting $k = 2$, 3rd term of $T_n = \frac{1}{n} f\left(\frac{2}{n}\right)$ represents area of 3rd strip as shown in graph as a_3

Putting $k = n - 1$, last term of $T_n = \frac{1}{n} f\left(\frac{n-1}{n}\right)$ represents area of last strip as shown in graph as a_n



So $\quad a_1 + a_2 + a_3 + ... + a_n > \int_0^1 \frac{1}{1 + x + x^2}\, dx = \frac{\pi}{3\sqrt{3}}$

So $\quad T_n > \frac{\pi}{3\sqrt{3}}$ **(D) is correct.**

9. A straight line through the vertex P of a triangle PQR intersects the side QR at the point S and the circumcircle of the triangle PQR at the point T. If S is not the centre of the circumcircle, then

(A) $\frac{1}{PS} + \frac{1}{ST} < \frac{2}{\sqrt{QS \times SR}}$

(B) $\frac{1}{PS} + \frac{1}{ST} > \frac{2}{\sqrt{QS \times SR}}$

(C) $\frac{1}{PS} + \frac{1}{ST} < \frac{4}{QR}$

(D) $\frac{1}{PS} + \frac{1}{ST} > \frac{4}{QR}$

Sol: Ans [D]

As from S, two chords of the circle are drawn,

$PS \cdot ST = SQ \cdot SR$ ...(i)

Now $\frac{1}{PS} + \frac{1}{ST} = \frac{PS + ST}{PS \cdot ST}$

Now $PS + ST \geq 2\sqrt{PS \cdot ST}$ (Equality holds if $PS = ST$, i.e. S is mid-point of PT)

$\Rightarrow \quad \frac{PS + ST}{\sqrt{PS \cdot ST}} \geq 2$

So $\quad \frac{1}{PS} + \frac{1}{ST} = \frac{PS + ST}{PS \cdot ST} = \frac{PS + ST}{\sqrt{PS \cdot ST}} \times \frac{1}{\sqrt{PS \cdot ST}}$

$$\geq \frac{2}{\sqrt{PS \cdot ST}} = \frac{2}{\sqrt{SQ \cdot SR}}$$ (Using (i))

So $\dfrac{1}{PS} + \dfrac{1}{ST} \geq \dfrac{2}{\sqrt{SQ \cdot SR}}$

Now $\sqrt{SQ \cdot SR} \leq \dfrac{SQ + SR}{2} = \dfrac{QR}{2}$ (Equality holds if $SQ = SR$, i.e., S is mid-point of QR)

$\Rightarrow \dfrac{1}{\sqrt{SQ \cdot SR}} \geq \dfrac{2}{QR}$

So $\dfrac{1}{PS} + \dfrac{1}{ST} \geq \dfrac{2}{\sqrt{SQ \cdot SR}} \geq \dfrac{4}{QR}$

Both equalities in last equation hold if S is mid-point of QR and PT both

$\Rightarrow$ Perpendicular bisectors of PT and QR will intersect in S.

$\Rightarrow$ S will become centre of circle, which is not given. So both equalities can't hold.

So, $\dfrac{1}{PS} + \dfrac{1}{ST} > \dfrac{4}{QR}$ **(D) is correct.**

10. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1 < 0$, $y_2 < 0$, be the end points of the latus rectum of the ellipse $x^2 + 4y^2 = 4$. The equations of parabolas with latus rectum PQ are

(A) $x^2 + 2\sqrt{3}\,y = 3 + \sqrt{3}$ (B) $x^2 - 2\sqrt{3}\,y = 3 + \sqrt{3}$

(C) $x^2 + 2\sqrt{3}\,y = 3 - \sqrt{3}$ (D) $x^2 - 2\sqrt{3}\,y = 3 - \sqrt{3}$

Sol: Ans [B,C]

Ellipse $x^2 + 4y^2 = 4$ can be written as $\dfrac{x^2}{4} + y^2 = 1$

$\Rightarrow a = 2, b = 1 \qquad \Rightarrow e^2 = 1 - \dfrac{b^2}{a^2} = 1 - \dfrac{1}{4} = \dfrac{3}{4}$

$\Rightarrow e = \dfrac{\sqrt{3}}{2}$

Find points of latus rectum are $P \equiv \left(-ae, \dfrac{-b^2}{a}\right) \equiv \left(-\sqrt{3}, \dfrac{-1}{2}\right)$ and $Q \equiv \left(ae, \dfrac{-b^2}{a}\right) \equiv \left(\sqrt{3}, \dfrac{-1}{2}\right)$

P, Q are end-points of latus rectum of parabola also.

So focus of parabola $\equiv$ mid-point of latus rectum PQ

$\equiv \left(0, \dfrac{-1}{2}\right)$

So axis of parabola is $x = 0$

Length of latus rectum = $4a = PQ = 2\sqrt{3}$

So vertex will lie at a distance of $a = \frac{\sqrt{3}}{2}$ from focus either in upward direction or downward direction.

So vertex could be $\left(0, \frac{\sqrt{3}-1}{2}\right)$ or $\left(0, -\left(\frac{\sqrt{3}+1}{2}\right)\right)$

So parabola could be $x^2 = 2\sqrt{3}\left(y + \frac{\sqrt{3}+1}{2}\right)$ or $x^2 = -2\sqrt{3}\left(y - \frac{\sqrt{3}-1}{2}\right)$

$\Rightarrow \quad x^2 - 2\sqrt{3}\,y = 3 + \sqrt{3}$ or $x^2 + 2\sqrt{3}\,y = 3 - \sqrt{3}$

(B) & (C) are correct.

SECTION- III

ASSERTION–REASON TYPE

This section contains 4 reasoning type questions. Each question has 4 choices (A), (B), (C) and (D) out of which ONLY ONE is correct.

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11. Consider three planes

$P_1 : x - y + z = 1$

$P_2 : x + y - z = -1$

$P_3 : x - 3y + 3z = 2.$

Let L_1, L_2, L_3 be the lines of intersection of the planes P_2 and P_3, P_3 and P_1, and P_1 and P_2, respectively.

STATEMENT-1: At least two of the lines L_1, L_2 and L_3 are non-parallel.

and

STATEMENT-2: The three planes do not have a common point.

(A) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1

(B) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1

(C) Statement-1 is True, Statement-2 is False

(D) Statement-1 is False, Statement-2 is True

Sol: Ans [D]

$P_1 : x - y + z = 1$

$P_2 : x + y - z = -1$

$P_3 : x - 3y + 3z = 2.$

dr's of lines L_1, L_2, L_3 are (0, –4, –4), (0, 2, 2) and (0, 2, 2)

As all lines are parallel because ratio of dr's are same.

$\Rightarrow$ Statement-1 is false

As $\Delta = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -3 & 3 \end{vmatrix} = 0$, but at least one of Δ_1, Δ_2 and Δ_3 is not zero.

$\Rightarrow$ No solution for given system

$\Rightarrow$ No common point

12. Consider the system of equations

$x - 2y + 3z = -1$

$-x + y - 2z = k$

$x - 3y + 4z = 1.$

STATEMENT-1: The system of equations has no solution for $k \neq 3$.

and

STATEMENT-2: The determinant $\begin{vmatrix} 1 & 3 & -1 \\ -1 & -2 & k \\ 1 & 4 & 1 \end{vmatrix} \neq 0$, for $k \neq 3$.

(A) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1

(B) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1

(C) Statement-1 is True, Statement-2 is False

(D) Statement-1 is False, Statement-2 is True

Sol: Ans [A]

$x - 2y + 3z = -1$

$-x + y - 2z = k$

$x - 3y + 4z = 1$

as $\begin{vmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = 0$

and $\begin{vmatrix} 1 & -1 & 3 \\ -1 & k & -2 \\ 1 & 1 & 4 \end{vmatrix} = (k - 3) \neq 0,$ for $k \neq 3$

$\Rightarrow$ Statement-1 and 2 are true and statement-2 is correct explanation

13. Let f and g be real valued functions defined on interval $(-1, 1)$ such that $g''(x)$ is continuous, $g(0) \neq 0$, $g'(0) = 0$, $g''(0) \neq 0$ and $f(x) = g(x) \sin x$.

STATEMENT-1: $\lim_{x \to 0} [g(x) \cot x - g(0) \operatorname{cosec} x] = f''(0)$.

and

STATEMENT-2: $f'(0) = g(0)$.

(A) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1

(B) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1

(C) Statement-1 is True, Statement-2 is False

(D) Statement-1 is False, Statement-2 is True

Sol: Ans [B]

$g(0) \neq 0,\ g'(0) = 0,\ g''(0) \neq 0,\ f(x) = g(x) \sin x$

$\lim_{x \to 0} (g(x) \cot x - g(0) \operatorname{cosec} x) = f''(0)$

Because

$f'(x) = g'(x)\sin x + \cos x\, g(x)$

$f''(x) = g''(x)\sin x + \cos x\, g'(x) + \cos x\, g'(x) - \sin x\, g(x)$

$f''(0) = 0 + g'(0) + g'(0) - 0 = 0$

Now,

$$\lim_{x \to 0} \frac{g(x) \cos x - g(0)}{\sin x} = \frac{g'(x) \cos x - \sin x\, g(x) - 0}{\cos x}$$

$$= \frac{g'(0) - 0}{1} = 0$$

Also $f'(0) = g(0)$

$\Rightarrow$ (B) is correct answer.

14. Consider the system of equations

$ax + by = 0,\ cx + dy = 0$, where $a, b, c, d \in \{0, 1\}$.

STATEMENT-1: The probability that the system of equations has a unique solution is 3/8.

and

STATEMENT-2: The probability that the system of equations has a solution is 1.

(A) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1

(B) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1

(C) Statement-1 is True, Statement-2 is False

(D) Statement-1 is False, Statement-2 is True

Sol: Ans [B]

$ax + by = 0, \ cx + dy = 0; \quad a, b, c, d \in \{0, 1\}$

For unique solution needed

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$$

$\Rightarrow \quad ad - bc \neq 0$

Total no. of possible determinants are = $2 \times 2 \times 2 \times 2 \quad \Rightarrow \quad 16$

No. of determinants such that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$ are 6

$\Rightarrow \quad P = \frac{6}{16} = \frac{3}{8}$

Obviously system of equation will always have a solution since $\Delta_x = \Delta_y$.

$\Rightarrow$ Both statement-1 and statement-2 are true, but statement-2 is not the correct explanation

SECTION- IV

LINKED COMPREHENSION TYPE

This section contains 3 Paragraphs. Based upon each paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

Paragraph for Question Nos. 15 to 17

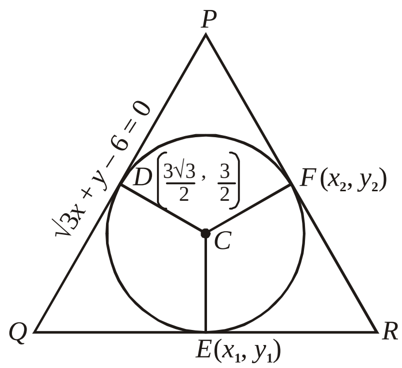
A circle *C* of radius 1 is inscribed in an equilateral triangle *PQR*. The points of contact of *C* with the sides *PQ*, *QR*, *RP* are *D*, *E*, *F*, respectively. The line *PQ* is given by the equation $\sqrt{3}\,x + y - 6 = 0$ and the point *D* is $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$. Further, it is given that the origin and the centre of *C* are on the same side of the line *PQ*.

15. The equation of circle *C* is

(A) $(x - 2\sqrt{3})^2 + (y - 1)^2 = 1$ (B) $(x - 2\sqrt{3})^2 + (y + 1/2)^2 = 1$

(C) $(x - \sqrt{3})^2 + (y + 1)^2 = 1$ (D) $(x - \sqrt{3})^2 + (y - 1)^2 = 1$

Sol: Ans [D]



Centre $\Rightarrow (h, k)$

as $\sqrt{3} \cdot 0 + 0 - 6 < 0$

$\Rightarrow \quad -\frac{(\sqrt{3}h + k - 6)}{\sqrt{3+1}} = 1$

$\Rightarrow \quad \sqrt{3}\,h + k - 6 = -2$

$\Rightarrow \quad \sqrt{3}\,h + k = 4$ ...(i)

Equation of *DC*

$x - \sqrt{3}\,y = k'$

$\frac{3\sqrt{3}}{2} - \sqrt{3}\,\frac{3}{2} = k'$

$\Rightarrow \quad k' = 0$

$\Rightarrow$ equation of *DC* is

$x - \sqrt{3}\,y = 0$

$h - \sqrt{3}\,k = 0$ ...(ii)

by (i) & (ii)

$h = \sqrt{3}\,,\ k = 1$

Equation of circle is $(x - \sqrt{3}\,)^2 + (y - 1)^2 = 1$

16. Points *E* and *F* are given by

(A) $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right), (\sqrt{3},\ 0)$

(B) $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), (\sqrt{3},\ 0)$

(C) $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

(D) $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

Sol: Ans [A]

$$\frac{(\sqrt{3} + i) - \left(\frac{3\sqrt{3}}{2} + i\,\frac{3}{2}\right)}{(\sqrt{3} + i) - (x_1 + iy_1)} = e^{i(2\pi/3)}$$

$\Rightarrow \quad E = (\sqrt{3}\,, 0)$

Similarly $\frac{(\sqrt{3} + i) - (\sqrt{3} + i0)}{(\sqrt{3} + i) - (x_2 + iy_2)} = e^{i(2\pi/3)}$

$\Rightarrow \quad F = \left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$

$\Rightarrow$ Points are $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right), (\sqrt{3},\ 0)$

17. Equation of the sides *QR*, *RP* are

(A) $y = \frac{2}{\sqrt{3}}x + 1,\ y = -\frac{2}{\sqrt{3}}x - 1$

(B) $y = \frac{1}{\sqrt{3}}x,\ y = 0$

(C) $y = \frac{\sqrt{3}}{2}x + 1,\ y = -\frac{\sqrt{3}}{2}x - 1$

(D) $y = \sqrt{3}\,x,\ y = 0$

Sol: Ans [D]

Slope of $EC = \frac{1-0}{\sqrt{3}-\sqrt{3}} = \infty$

$\Rightarrow$ QR is parallel to x-axis as point $E = (\sqrt{3}, 0)$

$\Rightarrow$ equation of QR is $y = 0$

Slope of $CF = \frac{1-3/2}{\sqrt{3}-\sqrt{3}/2} = \frac{-1/2}{\sqrt{3}/2} = -\frac{1}{\sqrt{3}}$

Slope of PR $\Rightarrow$ $m \times \left(-\frac{1}{\sqrt{3}}\right) = -1$

$\Rightarrow$ $m = \sqrt{3}$

$\Rightarrow$ equation of line is $y - \frac{3}{2} = \sqrt{3}\left(x - \frac{\sqrt{3}}{2}\right)$

$\Rightarrow$ $y = \sqrt{3}\,x$

$\Rightarrow$ $y = \sqrt{3}\,x,\ y = 0$

Paragraph for Question Nos. 18 to 20

Consider the functions defined implicitly by the equation $y^3 - 3y + x = 0$ on various intervals in the real line.

If $x \in (-\infty, -2) \cup (2, \infty)$, the equation implicitly defines a unique real valued differentiable function $y = f(x)$.

If $x \in (-2, 2)$, the equation implicitly defines a unique real valued differentiable function $y = g(x)$ satisfying $g(0) = 0$.

18. If $f(-10\sqrt{2}) = 2\sqrt{2}$, then $f''(-10\sqrt{2}) =$

(A) $\frac{4\sqrt{2}}{7^3 3^2}$ (B) $-\frac{4\sqrt{2}}{7^3 3^2}$ (C) $\frac{4\sqrt{2}}{7^3 3}$ (D) $-\frac{4\sqrt{2}}{7^3 3}$

Sol: Ans [B]

$y^3 - 3y + x = 0$

$$3y^2 \frac{dy}{dx} - 3\frac{dy}{dx} + 1 = 0 \qquad \text{...(i)}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{3y^2 - 3} = -\frac{1}{21}$$

From (i),

$$6y\left(\frac{dy}{dx}\right)^2 + 3y^2 \frac{d^2y}{dx^2} - 3\frac{d^2y}{dx^2} = 0$$

$$6 \times 2\sqrt{2} \times \frac{1}{(21)^2} + 3 \times 8\frac{d^2y}{dx^2} - 3\frac{d^2y}{dx^2} = 0$$

$$\frac{d^2y}{dx^2} = \frac{4\sqrt{2}}{7^3 3^2}$$

19. The area of the region bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$, where $-\infty < a < b < -2$, is

(A) $\int_a^b \frac{x}{3((f(x))^2 - 1)} dx + bf(b) - af(a)$

(B) $-\int_a^b \frac{x}{3((f(x))^2 - 1)} dx + bf(b) - af(a)$

(C) $\int_a^b \frac{x}{3((f(x))^2 - 1)} dx - bf(b) + af(a)$

(D) $-\int_a^b \frac{x}{3((f(x))^2 - 1)} dx - bf(b) + af(a)$

Sol: Ans [A] $y^3 - 3y + x = 0$

$$3y^2 \frac{dy}{dx} - 3\frac{dy}{dx} + 1 = 0$$

$$\Rightarrow \quad 3(y^2 - 1)dy + dx = 0$$

$$\text{Area} = \int_a^b y\,dx,$$

$$= b\,f(b) - a\,f(a) - \int_{f(a)}^{f(b)} x\,dy$$

$$= b\,f(b) - a\,f(a) + \int_a^b \frac{x}{3(y^2 - 1)} dx$$

20. $\int_{-1}^{1} g'(x)\ dx =$

(A) $2g(-1)$ (B) 0 (C) $-2g(1)$ (D) $2g(1)$

Sol: Ans []

$g(0) = 0$

$$\int_{-1}^{1} g'(x)\,dx = \int_{-1}^{1}\left(\frac{dy}{dx}\right)dx = \int_{-1}^{1} dy$$

$= y(1) - y(-1) = 2y(1)$

$= 2g(1)$

Put $x = 1$, $y^3 - 3y + 1 = 0$ ...(i)

$x = -1 \Rightarrow y^3 - 3y - 1 = 0$ ...(ii)

If α be the root of equation (i) then $(-\alpha)$ be the root of equation (ii)

$\Rightarrow y(-1) = -y(1)$

Paragraph for Question Nos. 21 to 23

Let A, B, C be three sets of complex numbers as defined below

$A = \{z : \text{Im}\,z \ge 1\}$

$B = \{z : |z - 2 - i| = 3\}$

$C = \{z : \text{Re}\,((1 - i)z) = \sqrt{2}\}$

21. The number of elements in the set $A \cap B \cap C$ is

(A) 0 (B) 1 (C) 2 (D) ∞

Sol: Ans [B]

The locus of B is the circle $(x - 2)^2 + (y - 1)^2 = 9$ ...(i)

The locus of C is the line $x + y = \sqrt{2}$

The locus of A is the region $y \ge 1$

Solving for $B \cap C$, we should get two points only (as circle and line can intersect in maximum 2 points)

Solving for $B \cap C$,

$x = \sqrt{2} - y$

Putting it in (i),

$(\sqrt{2} - y - 2)^2 + (y - 1)^2 = 9$

$\Rightarrow (y + 2 - \sqrt{2})^2 + (y - 1)^2 = 9$

$\Rightarrow 2y^2 + y(2 - 2\sqrt{2}) - 4\sqrt{2} - 2 = 0$

$\Rightarrow \quad y^2 + y(1 - \sqrt{2}) - 2\sqrt{2} - 1 = 0$

$\Rightarrow \quad y = \dfrac{\sqrt{2} - 1 \pm \sqrt{7 + 6\sqrt{2}}}{2}$

Now $\quad 7 + 6\sqrt{2} \geq 13$

$\Rightarrow \quad \sqrt{7 + 6\sqrt{2}} \geq 3$

So $\quad \dfrac{\sqrt{2} - 1 - \sqrt{7 + 6\sqrt{2}}}{2} < 0$ so rejected while taking intersection with A.

$\Rightarrow \quad y = \dfrac{\sqrt{2} - 1 + \sqrt{7 + 6\sqrt{2}}}{2} > \dfrac{3}{2}$ so selected while taking intersection with A.

So $A \cap B \cap C$ gives just one point.

22. Let z be any point in $A \cap B \cap C$. Then, $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between

(A) 25 and 29 (B) 30 and 34 (C) 35 and 39 (D) 40 and 44

Sol: Ans [C]

As we know that z is just one point satisfying $(x - 2)^2 + (y - 1)^2 = 9$ and $x + y = \sqrt{2}$ and $y \geq 1$

Now

$$|z + 1 - i|^2 + |z - 5 - i|^2 = |x + iy + 1 - i|^2 + |x + iy - 5 - i|^2$$
$$= (x + 1)^2 + (y - 1)^2 + (x - 5)^2 + (y - 1)^2$$
$$= 2x^2 - 8x + 26 + 2(y - 1)^2$$
$$= 2((x - 2)^2 + (y - 1)^2) + 18$$
$$= 2 \times 9 + 18$$
$$= 36$$

So lies between 35 nd 39.

23. Let z be any point is $A \cap B \cap C$ and let w be any point satisfying $|w - 2 - i| < 3$. Then, $|z| - |w| + 3$ lies between

(A) –6 and 3 (B) –3 and 6 (C) –6 and 6 (D) –3 and 9

Sol: Ans [B]

$\big||w| - |z + i|\big| \leq |w - 2 - i| < 3$

$\Rightarrow \quad \big||w| - \sqrt{5}\big| < 3$

$\Rightarrow \quad -3 < |w| - \sqrt{5} < 3$

$\Rightarrow \quad 0 \leq |w| < 3 + \sqrt{5}$

$\Rightarrow \quad -\sqrt{5} < 3 - |w| \leq 3$

Now

$(x-2)^2 + (y-1)^2 = 9$

$\Rightarrow \quad (y-1)^2 \leq 9$

$\Rightarrow \quad |y-1| \leq 3$

$\Rightarrow \quad -3 \leq y - 1 \leq 3$

$\Rightarrow \quad 1 \leq y \leq 4$

$(x-2)^2 + (y-1)^2 = 9$

$\Rightarrow \quad x^2 + y^2 = 4x + 2y + 4$

$= 4(x+y) + 4 - 2y$

$= 4\sqrt{2} + 4 - 2y \qquad (x + y = \sqrt{2})$

Now $\quad 1 \leq y \leq 4$

$\Rightarrow \quad -2 \geq -2y \geq -8$

$\Rightarrow \quad 9 > 4\sqrt{2} + 2 \geq 4\sqrt{2} + 4 - 2y \geq 4\sqrt{2} - 4 > 1$

So $\quad 1 < x^2 + y^2 < 9$

$\Rightarrow \quad 1 < \sqrt{x^2 + y^2} < 3$

$\Rightarrow \quad 1 < |z| < 3$ and $-\sqrt{5} < 3 - |w| \leq 3$

$\Rightarrow \quad -3 < 1 - \sqrt{5} < |z| + 3 - |w| < 6$