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Joint Economic Production - Inventory Model of Single-Manufacturer-Single-Buyer for Time Dependent Deteriorating Items with Imperfect Quality When Demand is Quadratic

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An integrated economic production-ordering inventory model is developed for single-manufacturer-single-buyer. The units in inventory of both the players are subject to deterioration with time. Two parameters Weibull distribution is considered for deterioration. The manufacturer faces problem of producing poor quality items during production which needs rework. The multiple deliveries by the buyer are considered. The model is analyzing a market where the demand of the product is increasing quadratically with time. The joint total cost is minimized using a classical optimization technique. A numerical example is given to support the proposed mathematical development. The sensitivity analysis is carried out to study the variations in the decision variables and objective function. It is established that the joint decision results in lowering the joint total cost compared with an independent decision by the manufacturer and the buyer.

Introduction

Deterioration is defined as spoilage, vaporization, damage, or breakage which results in a loss of utility or loss of marginal value and hence resulting in the decrease in the utility. The products like pharmaceutical drugs, radioactive chemicals, fruits and vegetables, sea-food deteriorate with time, when stocked in the ware-house. Ghare and Schrader (1963) considered Weibull distribution to describe time dependent deterioration and derived an inventory model. For the literature on deteriorating inventory one can refer to literature surveys by Nahmias (1982), Raafat (1991), Shah and Shah (2000), Goyal and Giri (2001). Most of the article deals with the single player of supply chain. However, in competitive market, the entrepreneurs started working in a supply chain with their limited resources. This helps them to work efficiently and up to the maximum service level of the customer.

Goyal and Nebebe (2000) developed the optimal production and shipment policy for a single-vendor-singer-buyer. Woo et al. (2001) derived a joint integrated policy when a manufacturer purchases raw material from outside resources to produce items delivers the finished goods to the multiple buyers. Rau et al. (2003) formulated a multi-echelon inventory model when units in inventory are subject to constant rate of deterioration to derive an optimal joint total cost from the integration point of view among the supplier, the manufacturer and the buyer. Yang and Wee (2003) modeled an integrated inventory model with constant rate of deterioration and multiple deliveries. Shah et al. (2008) considered salvage value for deteriorating items in Yang and Wee (2003) model. Hans et al. (2006) gave another approach to obtain the joint economic lot-size in distribution system with multiple shipment policy.

Due to faulty machine, poor quality of raw material, mishandling, the produced items may not be of perfect quality. After inspection it may require some rework. The concept of imperfect production process for deteriorating items was explored by Kim and Hong (1999) to obtain optimal production time. Salameh and Jaber (2000) derived optimum production and ordering strategies for items with imperfect quality. They assumed that the poor-quality items are disposed only in a single batch after the screening. Goyal et al. (2002) extended above idea to compute the optimal production

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quantity for items with imperfect quality. Chang and Hou (2003) extended above concept for deteriorating production inventory system by allowing shortages. Papachristos and Konstantaras (2006) developed the economic ordering inventory model for the items with imperfect quality.

The demand of products like seasonal paddy grains, fashion-apparels etc. increases exponentially with time. In this article, an integrated production-ordering inventory model with a time varying rate of deterioration under imperfect production processes and partial backordering is analyzed when demand is increasing quadratically with time. The model is supported with a numerical example. The sensitivity analysis is carried out for all the parameters and searched for the critical parameters. The percentage savings in the total integrated cost with joint and independent solution is presented.

Notations and Assumptions:

The mathematical model is developed using the following notations and assumptions:

Notations:

$R(t)$ Buyer's demand rate (units/unit time) = $a(1 + bt + ct^2)$, where a is fixed demand, $0 < b, c < 1$ denotes linear and exponential rate of change of demand with respect to time.

$P(t)$ $= \gamma R(t)$; production rate(units/unit time), where $\gamma > 1$

B Fraction of buyer's demand back-ordered

N The number of delivery per order

T_1 The production period

T_2 The non-production period

T_3 The period for which the buyer carries positive stock

T_4 The period for which a buyer suffers back-logging

T $(= T_1 + T_2)$; the length of the cycle time

$I_{rm}(t1)$ Raw material's inventory level at any time t_1, $0 \le t_1 \le T_1$

$I_{mfi}(ti)$ Manufacturer's finished goods inventory level at any time ti, $0 \le ti \le Ti$, i = 1,2

$I_{ri}(ti)$ Buyer's inventory level at any time ti, $0 \le ti \le Ti$, i = 3, 4

$\theta_i(t) = \alpha_i \beta_i, t^{\beta i-1}$ i = rm, m, r, where αi denotes scale parameter, β_i denotes shape parameter and t denotes time. $0 < \alpha_i < 1$ and $\beta i > 1$

Note: $\alpha_{rm} < \alpha_m < \alpha_r$ and $\beta_{rm} < \beta_m < \beta_r$

A_{mo} Manufacturer's ordering cost per order cycle ($/cycle)

A_{ms} Manufacturer's set-up cost per production cycle ($/cycle)

A_r Buyer's ordering cost per order cycle ($/order)

C_{mr} Manufacturer's raw material purchase cost per unit cost ($/unit)

C_{mrw} Manufacturer's finished goods rework cost per unit ($/unit)

C_{mf} Manufacturer's finished goods per unit cost ($/unit)

C_r Buyer's purchase cost per unit ($/unit)

h_{mr} Manufacturer's raw material inventory holding cost per unit per time unit ($/unit/unit time)

h_{mf} Manufacturer's finished goods inventory holding cost per unit per unit time ($/unit/unit time)

h_r Buyer's inventory holding cost per unit per unit time ($/unit/unit time)

π_{rB} Buyer's backlog cost per unit per unit time ($/unit/unit time)

π_{rL} Buyer's per unit lost sale cost per unit time ($/unit/unit time)

M_{Im} maximum inventory level of finished goods of the manufacturer

M_{Ir} maximum inventory level of buyer

Q_{rm} Raw material's order quantity per order

Q_m Manufacturer's finished goods production units per production

Q_r Buyer's received quantity per delivery from the manufacturer

X Random elapsed time until production process shift

f(X) Probability distribution function of X

D Percentage of defective items produced once system is in the out-of-control state

TC_{rm} Total cost of the raw material per unit time

TC_m Total cost of the manufacturer per unit time

TC_r Total cost of buyer per unit time

TC Total joint cost of an inventory system per unit time

Assumptions:

1. The inventory system under consideration deals with a single item.
2. Single-manufacturer and single-buyer are stackers of supply chain.
3. Demand rate is quadratic and increasing function of time. Production rate is proportional to the demand rate.
4. Shortages at the manufacturer side are not allowed. Replenishment is instantaneous.
5. Partial backlogging is allowed only for the buyer. It is cleared from the arrival of the next replenishment.
6. Multiple deliveries per order are considered. The planning horizon is infinite and the cycles during the planning horizon are continuous. The items of the first delivery are made in the previous cycle.
7. In the beginning of each production cycle, items produced are of good quality i.e. the production process is in in-control state.
8. During the production phase, the production process may shift from an in-control state to an out-of-control state. An elapsed time is exponentially distributed with the known mean. Once the production process shifts to an out-of-control state, the shift can not be detected only at the end of the production cycle.
9. A percentage of the produced items are defective. There is a rework cost associated to the defective items.
10. The units in inventory are subject to deteriorate with time. The deterioration rate follows two-parameter Weibull distribution. The deteriorated units can neither be repaired nor replaced during the cycle time under consideration.

Mathematical Model

One needs to analyze stages of the joint venture of manufacturer- buyer. The first stage is the manufacturer's production and non-production phases. The manufacturer procures raw material from outer supplier to produce finished goods. The second stage is of buyer's inventory level. The buyer purchases units in multiple deliveries during cycle time.

- **Sub-system for raw-material inventory**

The status of manufacturer's raw material's inventory is shown in Fig. 1. The manufacture procures Qrm - units in the beginning of the cycle. During T1 – time, the inventory level of the raw-material depletes due to the manufacturer's demand and the deterioration of units with time of the raw material.

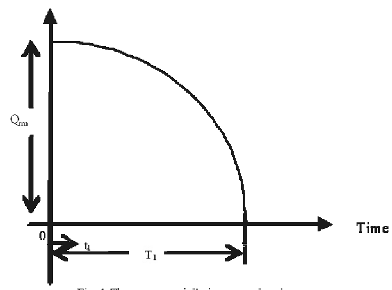


Fig. 1 The raw material's inventory level.

The rate of change of the inventory level at any instant of time t1 $(0 \le t1 \le T1)$ is given by the differential equation:

$$\frac{dI_{rm}(t_1)}{dt_1} + \theta rm(t1)Irm(t1) = -P(t1), \quad 0 \le t1 \le T1$$

Equivalently,

$$\frac{dI_{rm}(t_1)}{dt_1} + \theta rm(t1)Irm(t1) = -\gamma R(t1), \quad 0 \le t1 \le T1 \tag{3.1}$$

Under the assumption that $0 < \alpha_{rm} < 1$, using series expansion of exponential function and neglecting α^2_{rm} and its higher powers, the solution of differential equation with $I_{rm}(T_1) = 0$ is given by

$I_{rm}(t1)=$

$$-\gamma a t_1+\gamma a T_1+\frac{1}{2}\gamma a b T_1^2+\frac{1}{3}\gamma a c T_1^3-\frac{1}{2}\gamma a b t_1^2-\frac{1}{3}\gamma a c t_1^3+\gamma a t_1 \alpha_{rm} t_1^{\beta_{rm}}$$
$$-\gamma a T_1 \alpha_{rm} t_1^{\beta_{rm}}-\frac{1}{2}\gamma a b T_1^2 \alpha_{rm} t_1^{\beta_{rm}}-\frac{1}{3}\gamma a c T_1^3 \alpha_{rm} t_1^{\beta_{rm}}+\frac{1}{2}\gamma a b t_1^2 \alpha_{rm} t_1^{\beta_{rm}}$$
$$+\frac{1}{3}\gamma a c t_1^3 \alpha_{rm} t_1^{\beta_{rm}}+\frac{\gamma a \alpha_{rm} T_1^{\beta_{rm}} T_1}{\beta_{rm}+1}+\frac{\gamma a \alpha_{rm} T_1^{\beta_{rm}} T_1^2 b}{\beta_{rm}+1}$$
$$+\frac{\gamma a \alpha_{rm} T_1^{\beta_{rm}} T_1^3 c}{\beta_{rm}+1}-\frac{\gamma a \alpha_{rm} T_1^{\beta_{rm}} T_1^2 b}{(\beta_{rm}+1)(\beta_{rm}+2)}-\frac{2\gamma a \alpha_{rm} T_1^{\beta_{rm}} T_1^3 c}{(\beta_{rm}+1)(\beta_{rm}+2)}$$
$$+\frac{2\gamma a c \alpha_{rm} T_1^{\beta_{rm}} T_1^3}{(\beta_{rm}+1)(\beta_{rm}+2)(\beta_{rm}+3)}-\frac{\gamma a \alpha_{rm} t_1^{\beta_{rm}} t_1}{\beta_{rm}+1}-\frac{\gamma a \alpha_{rm} t_1^{\beta_{rm}} t_1^2 b}{\beta_{rm}+1}$$
$$-\frac{\gamma a \alpha_{rm} t_1^{\beta_{rm}} t_1^3 c}{\beta_{rm}+1}+\frac{\gamma a \alpha_{rm} t_1^{\beta_{rm}} t_1^2 b}{(\beta_{rm}+1)(\beta_{rm}+2)}+\frac{2\gamma a \alpha_{rm} t_1^{\beta_{rm}} t_1^3 c}{(\beta_{rm}+1)(\beta_{rm}+2)}$$
$$-\frac{2\gamma a c \alpha_{rm} t_1^{\beta_{rm}} t_1^3}{(\beta_{rm}+1)(\beta_{rm}+2)(\beta_{rm}+3)}$$

$$0 \le t_1 \le T_1 \qquad (3.2)$$

The maximum inventory level of raw material is

$Q_{rm} = I_{rm}(0)$

$Q_{rm}=$

$$\gamma a T_1+\frac{1}{2}\gamma a b T_1^2+\frac{1}{3}\gamma a c T_1^3+\frac{\gamma a \alpha_{rm} T_1^{\beta_{rm}} T_1}{\beta_{rm}+1}+\frac{\gamma a \alpha_{rm} T_1^{\beta_{rm}} T_1^2 b}{\beta_{rm}+1}$$
$$+\frac{\gamma a \alpha_{rm} T_1^{\beta_{rm}} T_1^3 c}{\beta_{rm}+1}-\frac{\gamma a \alpha_{rm} T_1^{\beta_{rm}} T_1^2 b}{(\beta_{rm}+1)(\beta_{rm}+2)}-\frac{2\gamma a \alpha_{rm} T_1^{\beta_{rm}} T_1^3 c}{(\beta_{rm}+1)(\beta_{rm}+2)}$$
$$+\frac{2\gamma a c \alpha_{rm} T_1^{\beta_{rm}} T_1^3}{(\beta_{rm}+1)(\beta_{rm}+2)(\beta_{rm}+3)}$$

(3.3)

The cost-components of the raw material inventory system are as follows:

- The ordering cost; OCrm of raw material per order is

$$OC_{rm} = A_{mo} \qquad (3.4)$$

- The purchase cost; PCrm of raw material is

$$PC_{rm} = C_{mr} Q_{rm} \qquad (3.5)$$

- The manufacturer's raw material inventory holding cost is

$$IHC_{rm} = h_{mr} \int_0^{T_1} I_{rm}(t_1)dt_1 \qquad (3.6)$$

Hence, for the raw material, total cost per time unit is

$$TC_{rm} = \frac{1}{T}\,[OC_{rm} + PC_{rm} + IHC_{rm}] \qquad (3.7)$$

Next, we study the manufacturer's finished goods inventory system. The depletion of inventory is graphed in Fig 2.

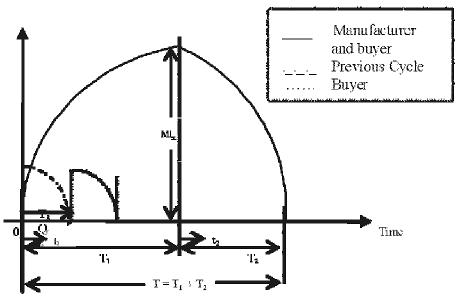


Fig. 2 The manufacturer inventory level

For manufacturer, there are two phases viz. production phase and non-production phase. The production phase starts at t1 = 0 and continues till maximum inventory MIm is produced at t1 = T1. During T1- time period, the inventory depletes due to demand during production phase and deterioration of units at the time dependent rate; θm(t1). At T1-time, production stops and there after the depletion of manufacturer's inventory level is due to the buyer's demand and deterioration of units. The inventory level reaches to zero at t2 = T2. The inventory level at any instant of time during production and non-production phases can be described by the following differential equations:

$$\frac{dI_{mf_1}(t_1)}{dt_1} = P(t1) - R(t1) - \theta m(t1) Imf1(t1)$$

$$= (\gamma-1)R(t_1) - \theta_m(t_1)I_{mf1}(t_1),\ 0 \le t_1 \le T_1 \quad (3.8)$$

and

$$\frac{dI_{mf_2}(t_2)}{dt_2} = -R(t_2) - \theta_m(t_2)I_{mf2}(t_2),\ 0 \le t_2 \le T_2 \quad (3.9)$$

Using the boundary conditions $I_{mf1}(0) = 0$ and $I_{mf2}(T_2) = 0$, the solution of the differential equations (3.8) and (3.9) can be given by

$$I_{mf1}(t_1) = \gamma a t_1 - a t_1 - \frac{a\alpha_m t_1^{\beta_m} t_1^2 b}{\beta_m+1} - \frac{a\alpha_m t_1^{\beta_m} t_1^3 c}{\beta_m+1} + \frac{1}{2} a b t_1^2 \alpha_m t_1^{\beta_m} - \frac{a\alpha_m t_1^{\beta_m} t_1}{\beta_m+1}$$
$$-\frac{1}{3} a c t_1^3 + \frac{a\alpha_m t_1^{\beta_m} t_1^2 b}{(\beta_m+1)(\beta_m+2)} + \frac{2a\alpha_m t_1^{\beta_m} t_1^3 c}{(\beta_m+1)(\beta_m+2)} + \frac{a\gamma\alpha_m t_1^{\beta_m} t_1^2 b}{\beta_m+1}$$
$$+\frac{a\gamma\alpha_m t_1^{\beta_m} t_1^3 c}{\beta_m+1} - a\gamma t_1 \alpha_m t_1^{\beta_m} + \frac{a\gamma\alpha_m t_1^{\beta_m} t_1}{\beta_m+1} + \frac{1}{3} a c t_1^3 \alpha_m t_1^{\beta_m}$$
$$-\frac{2ac\alpha_m t_1^{\beta_m} t_1^3}{(\beta_m+1)(\beta_m+2)(\beta_m+3)} - \frac{1}{2} a\gamma b t_1^2 \alpha_m t_1^{\beta_m} + \frac{2a\gamma c\alpha_m t_1^{\beta_m} t_1^3}{(\beta_m+1)(\beta_m+2)(\beta_m+3)}$$
$$-\frac{1}{3} a\gamma c t_1^3 \alpha_m t_1^{\beta_m} - \frac{2a\gamma\alpha_m t_1^{\beta_m} t_1^3 c}{(\beta_m+1)(\beta_m+2)} - \frac{a\gamma\alpha_m t_1^{\beta_m} t_1^2 b}{(\beta_m+1)(\beta_m+2)} + \frac{1}{2}\gamma a b t_1^2$$
$$+\frac{1}{3}\gamma a c t_1^3 + a t_1 \alpha_m t_1^{\beta_m} - \frac{1}{2} a b t_1^2, \quad 0 \le t_1 \le T_1 \quad (3.10)$$

and

$$I_{mf2}(t_2) = -a T_2 \alpha_m t_2^{\beta_m} + a t_2 \alpha_m t_2^{\beta_m} - \frac{1}{3} a c t_2^3 + \frac{1}{3} a c t_2^3 \alpha_m t_2^{\beta_m} + \frac{a\alpha_m T_2^{\beta_m} T_2}{\beta_m+1} - a t_2 + a T_2$$
$$+\frac{1}{2} a b T_2^2 + \frac{1}{3} a c T_2^3 + \frac{a\alpha_m T_2^{\beta_m} T_2^2 b}{\beta_m+1} + \frac{a\alpha_m T_2^{\beta_m} T_2^3 c}{\beta_m+1} - \frac{1}{2} a b t_2^2$$
$$-\frac{2a\alpha_m T_2^{\beta_m} T_2^3 c}{(\beta_m+1)(\beta_m+2)} - \frac{a\alpha_m T_2^{\beta_m} T_2^2 b}{(\beta_m+1)(\beta_m+2)} + \frac{2ac\alpha_m T_2^{\beta_m} T_2^3}{(\beta_m+1)(\beta_m+2)(\beta_m+3)}$$
$$-\frac{a\alpha_m t_2^{\beta_m} t_2^2 b}{\beta_m+1} - \frac{a\alpha_m t_2^{\beta_m} t_2^3 c}{\beta_m+1} + \frac{a\alpha_m t_2^{\beta_m} t_2^2 b}{(\beta_m+1)(\beta_m+2)} - \frac{a\alpha_m t_2^{\beta_m} t_2}{\beta_m+1}$$
$$+\frac{2a\alpha_m t_2^{\beta_m} t_2^3 c}{(\beta_m+1)(\beta_m+2)} - \frac{1}{3} a c T_2^3 \alpha_m t_2^{\beta_m} + \frac{1}{2} a b t_2^2 \alpha_m t_2^{\beta_m}$$
$$-\frac{2ac\alpha_m t_2^{\beta_m} t_2^3}{(\beta_m+1)(\beta_m+2)(\beta_m+3)} - \frac{1}{2} a b T_2^2 \alpha_m t_2^{\beta_m}, \quad 0 \le t2 \le T2 \quad (3.11)$$

respectively. The maximum inventory of manufacturer is

$$MI_m = I_{mf2}(0) = \tag{3.12}$$

$$\frac{a\alpha_m T_2^{\beta_m} T_2}{\beta_m+1} + aT_2 + \frac{1}{2}abT_2^2 + \frac{1}{3}acT_2^3 + \frac{a\alpha_m T_2^{\beta_m} T_2^2 b}{\beta_m+1} + \frac{a\alpha_m T_2^{\beta_m} T_2^3 c}{\beta_m+1} - \frac{2a\alpha_m T_2^{\beta_m} T_2^3 c}{(\beta_m+1)(\beta_m+2)} - \frac{a\alpha_m T_2^{\beta_m} T_2^2 b}{(\beta_m+1)(\beta_m+2)} + \frac{2ac\alpha_m T_2^{\beta_m} T_2^3}{(\beta_m+1)(\beta_m+2)(\beta_m+3)}$$

The units produced during $[0, T_1]$ is

$$Q_m = P(T_1)\,T_1 = a\gamma T_1[1+bT_1 + cT_1^2] \tag{3.13}$$

The various cost components of manufacturer's inventory system are as follows:

- The initial production set-up cost per cycle is

$$OC_m = A_{ms} \tag{3.14}$$

- The inventory is stored in warehouse during T1 and T2 -time periods. If this system does not consider the buyer, all of inventory holding cost is of the manufacturer, which are first two terms in equation (3.15). If this system considers the buyer, the inventory holding cost of the items which are supplied to the buyer is to be subtracted from that of the manufacturer which is the third term in the equation (3.15). Hence, the Manufacturer's inventory holding cost is

$$IHC_m = h_{mf}\int_0^{T_1} I_{mf1}(t_1)dt_1 + h_{mf}\int_0^{T_2} I_{mf2}(t_2)dt_2 - h_{mf}\int_0^{T_3} I_{r3}(t_3)dt_3$$

- The production cost of the manufacturer is

$$PC_m = C_{mf}\,Q_m \tag{3.16}$$

- The number of defective items; N in a production cycle is given by

$$N = \begin{cases} 0 & ,\text{when } X \ge T_1 \\ dP(T_1)(T_1 - X) & ,\text{when } X < T_1 \end{cases} \tag{3.17}$$

Then, the expected number of defective items during a production cycle is

$$E(N) = \int_0^{T_1} Nf(X)dX \tag{3.18}$$

Under the assumption that an elapsed time shift is exponentially distributed with a mean of $\frac{1}{\mu}$, the rework cost is

$$RW = C_{mrw}\, d\, P(T_1)\, \mu\int_0^{T_1}(T_1 - X)e^{-\mu X}dX \tag{3.19}$$

Therefore, the total cost per time unit of manufacturer inventory system is

$$TCm = \frac{1}{T}[OC_m + PC_m + IHC_m + RW] \tag{3.20}$$

Next, one needs to analyze the buyer's inventory system. The inventory level status of the buyer is exhibited in Fig. 3.

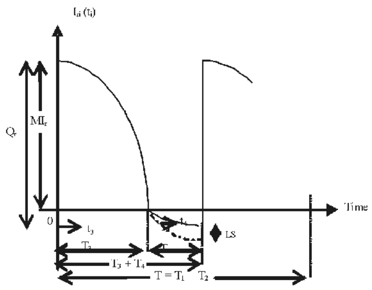


Fig. 3 The inventory status of the buyer

The buyer starts with a balance of MIr – units after clearing shortages at t3 = 0. During T3 – time period, buyer's inventory depletes due to the demand and deterioration of units. At T3 - time, the inventory level of buyer reaches zero. Then till T4 – time period, buyer faces shortages of which is partially is lost sales. Only the back-logged units are cleared from the next delivery. There are n -deliveries during the cycle time T (= T1 + T2). The buyer's inventory level at any instant of time can be described by the differential equations as:

$$\frac{dI_{r3}(t_3)}{dt_3} = -R(t_3)) - \theta_r I_{r3}(t_3), \tag{3.21}$$

and

$$\frac{dI_{r4}(t_4)}{dt_4} = -BR(t_4)\,,\ 0 \le t_4 \le T_4 \quad (3.22)$$

Using the boundary conditions $I_{r3}(T3) = 0$ and $I_{r4}(0) = 0$, the solutions of the differential equations are

$I_{r3}(t_3)=$

$$\begin{aligned}
&-a T_3 \alpha_r t_3^{\beta_r} + a t_3 \alpha_r t_3^{\beta_r} - \frac{a \alpha_r t_3^{\beta_r} t_3}{\beta_r+1} - \frac{a \alpha_r t_3^{\beta_r} t_3^2 b}{\beta_r+1} - \frac{a \alpha_r t_3^{\beta_r} t_3^3 c}{\beta_r+1} \\
&+ \frac{a \alpha_r t_3^{\beta_r} t_3^2 b}{(\beta_r+1)(\beta_r+2)} + \frac{2 a \alpha_r t_3^{\beta_r} t_3^3 c}{(\beta_r+1)(\beta_r+2)} + \frac{a \alpha_r^2 (t_3^{\beta_r})^2 t_3^3 c}{\beta_r+1} - \frac{a \alpha_r^2 (t_3^{\beta_r})^2 t_3^2 b}{(\beta_r+1)(\beta_r+2)} \\
&- \frac{2 a \alpha_r^2 (t_3^{\beta_r})^2 t_3^3 c}{(\beta_r+1)(\beta_r+2)} + \frac{2 a c \alpha_r^2 (t_3^{\beta_r})^2 t_3^3}{(\beta_r+1)(\beta_r+2)(\beta_r+3)} - \frac{2 a c \alpha_r t_3^{\beta_r} t_3^3}{(\beta_r+1)(\beta_r+2)(\beta_r+3)} \\
&- \frac{a \alpha_r^2 T_3^{\beta_r} T_3 t_3^{\beta_r}}{\beta_r+1} - \frac{a \alpha_r^2 T_3^{\beta_r} T_3^2 b\, t_3^{\beta_r}}{\beta_r+1} - \frac{a \alpha_r^2 T_3^{\beta_r} T_3^3 c\, t_3^{\beta_r}}{\beta_r+1} \\
&+ \frac{a \alpha_r^2 T_3^{\beta_r} T_3^2 b\, t_3^{\beta_r}}{(\beta_r+1)(\beta_r+2)} + \frac{1}{3} a c T_3^3 - \frac{1}{2} a b t_3^2 - \frac{1}{3} a c t_3^3 + \frac{1}{2} a b T_3^2 + a T_3 - a t_3 \\
&- \frac{1}{2} a b T_3^2 \alpha_r t_3^{\beta_r} - \frac{1}{3} a c T_3^3 \alpha_r t_3^{\beta_r} + \frac{1}{2} a b t_3^2 \alpha_r t_3^{\beta_r} + \frac{1}{3} a c t_3^3 \alpha_r t_3^{\beta_r} \\
&+ \frac{a \alpha_r^2 (t_3^{\beta_r})^2 t_3}{\beta_r+1} + \frac{2 a \alpha_r^2 T_3^{\beta_r} T_3^3 c\, t_3^{\beta_r}}{(\beta_r+1)(\beta_r+2)} - \frac{2 a c \alpha_r^2 T_3^{\beta_r} T_3^3 t_3^{\beta_r}}{(\beta_r+1)(\beta_r+2)(\beta_r+3)} \\
&+ \frac{a \alpha_r^2 (t_3^{\beta_r})^2 t_3^2 b}{\beta_r+1} + \frac{a \alpha_r T_3^{\beta_r} T_3}{\beta_r+1} + \frac{a \alpha_r T_3^{\beta_r} T_3^2 b}{\beta_r+1} + \frac{a \alpha_r T_3^{\beta_r} T_3^3 c}{\beta_r+1} \\
&- \frac{a \alpha_r T_3^{\beta_r} T_3^2 b}{(\beta_r+1)(\beta_r+2)} - \frac{2 a \alpha_r T_3^{\beta_r} T_3^3 c}{(\beta_r+1)(\beta_r+2)} + \frac{2 a c \alpha_r T_3^{\beta_r} T_3^3}{(\beta_r+1)(\beta_r+2)(\beta_r+3)}
\end{aligned}$$

$, 0 \le t_3 \le T_3$ (3.23)

and

$$I_{r4}(t_4) = -Ba\left[t_4 + \frac{b}{2} t_4^2 + \frac{c}{3} t_4^3\right], \quad (3.24)$$

$0 \le t_4 \le T_4$

The buyer's maximum inventory is given by

$MI_r = I_{r3}(0) =$

$$\begin{aligned}
&\frac{1}{3} a c T_3^3 + \frac{1}{2} a b T_3^2 + a T_3 + \frac{a \alpha_r T_3^{\beta_r} T_3}{\beta_r+1} + \frac{a \alpha_r T_3^{\beta_r} T_3^2 b}{\beta_r+1} + \frac{a \alpha_r T_3^{\beta_r} T_3^3 c}{\beta_r+1} \\
&- \frac{a \alpha_r T_3^{\beta_r} T_3^2 b}{(\beta_r+1)(\beta_r+2)} - \frac{2 a \alpha_r T_3^{\beta_r} T_3^3 c}{(\beta_r+1)(\beta_r+2)} + \frac{2 a c \alpha_r T_3^{\beta_r} T_3^3}{(\beta_r+1)(\beta_r+2)(\beta_r+3)}
\end{aligned}$$

(3.25)

Hence, the quantity to be purchased per delivery in the beginning of the buyer's cycle is

$$Qr = MI_r - BR(T_4)T_4 \quad (3.26)$$

The different costs components incurred by the buyer for n - deliveries are as follows:

- Ordering cost for n-orders is

  $OC_r = n\, A_r$ (3.27)

- Purchase cost for n-deliveries is

  $PC_r = n\, C_r Q_r$ (3.28)

- Inventory holding cost for n-replenishments is

  $IHC_r = n\, h_r \int_0^{T_1} I_{r3}(t_3)\,dt_3$ (3.29)

- Shortages cost for n-deliveries is

$$SC = n\ \pi_{rB} \int_0^{T_4} - I_{r4}\ (t_4)dt_4 \qquad (3.30)$$

- Lost sales cost for n-replenishments is

$$LS = n\ \pi_{rL} \int_0^{T_4} (1\text{-}B)R\ (t_4)dt_4 \qquad (3.31)$$

Hence, the buyer's total cost per time unit for n-deliveries is

$$TC_r = 1/T\ [\ OC_r + PC_r + IHC_r + SC + LS \qquad (3.32)$$

Using costs equation (3.7), (3.20) and (3.32) the total joint cost per unit time of an inventory system is

$$TC = TC_{rm} + TC_m + TC_r \qquad (3.33)$$

The total joint cost; TC is a function of discrete variable 'n' and continuous variables T_1, T_2, T_3 and T_4. Since $T= T_1 + T_2 = n\ (T_3 + T_4)$, we have

$$T_3 = \frac{T}{n} - T_4 \qquad (3.34)$$

Using the continuity of the functions $I_{mf1}(t_1)$ and $I_{mf2}(t_2)$ at T_1, we get

$$T_1 \approx \frac{1}{\nu - 1}\left(T_2 + \frac{bT_2^2}{2} + \frac{cT_2^3}{3} + \frac{\alpha_m T_2^{\beta_m + 1}}{\beta_m + 1}\right) \qquad (3.35)$$

With (3.34) and (3.35), (3.33), i.e. total integrated cost is function of the discrete variable; n and continuous variables T2 and T4.

Computational steps

To obtain minimum total joint cost perform following steps:

Step 1: Start with n = 1.

Step 2: Compute T_2 and T_4 by solving

$$\frac{\partial TC}{\partial T_2} = 0 \text{ and } \frac{\partial TC}{\partial T_4} = 0$$

Step 3: Compute TC using equation (3.33) with obtained values of T_2 and T_4.

Step 4: Increment n by 1 and repeat steps 2 and step 3 until the condition

$TC\ (n\text{-}1,\ T_2(n\text{-}1),\ T_4(n\text{-}1)\ ) \geq TC\ (n,\ T_2,\ T_4) \leq TC\ (n+1,\ T_2(n+1),\ T_4(n+1))$ is satisfied.

Step 5: Tabulate T_1, T_3, Q_{rm}, Q_m, Q_r, TC_{rm}, TC_m and TC_r.

Numerical Example:

Consider following parametric values in an appropriate units:

$[\ a, b, c, \gamma, B\ A_{mo}, A_{ms}, A_r, C_{mr}, C_{mf}, C_r, C_{mrw}, h_{mr}, h_{mf}, h_r, \pi_{rB}, \pi_{rL}, d, \mu, \alpha_{rm}, \alpha_{mf}, \alpha_r, \beta_{rm}, \beta_{mf}, \beta_r]$

= [750, 10%, 15%, 3, 0.8, 120, 90,60, 6, 9, 11, 10, 0.6, 0.8, 1.00, 10, 5, 5%, 0.001, 5%, 7%, 12%,1.2,1.7,2] The computational results are exhibited in Table 1.

Table 1: Computational results

n	T1	T2	T3	T4	Qrm	Qm	Qr	TCrm	TCm	TCr	TC
1	0.122	0.241	0.211	0.152	278	279	254	4941	7197	8402	20540
2	0.139	0.275	0.078	0.129	318	320	138	4917	7237	8313	20467
3	0.149	0.293	0.027	0.120	340	342	94	4909	7248	8303	20460
4	0.156	0.306	0.0002	0.116	356	359	70	4904	7255	8312	20470
5	0.162	0.318	-0.016	0.1126	369	372	56	4901	7261	8327	20488

Note: Grey region is the minimum total joint cost of an inventory system.
Blue solution is optimum from manufacturer end and green solution is optimum from raw material point.

Following conclusions are drawn for the data under consideration:

1. Three orders by the buyer gives minimum total joint cost; TC* = $20460. For this minimum cost, the manufacturer's production time T1 = 0.149 years and non-production time T2 = 0.293 years. The buyer stocks for time T3 = 0.027 years and has shortages for T4 =0.120 years. The total cycle time of manufacturer is 0.442 years and that of buyer is 0.147 years. The manufacturer produces 342 - units and ships 94 - units in each delivery. The convexity of the total joint cost TC with respect to T2 and T4 is shown in Fig. 4 and with respect to n is shown in Fig. 5.

Fig. 5 Convexity of total joint cost w. r. t. n for T_2 = 0.293 and T_4 = 0.120

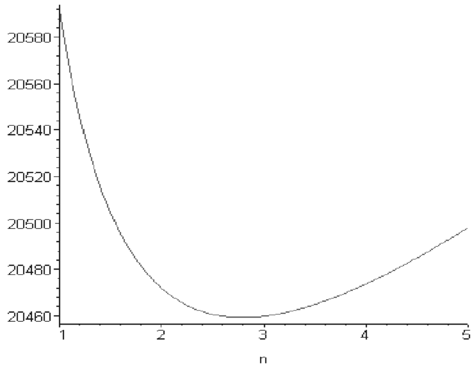
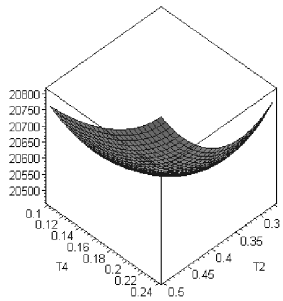


Fig.4 Convexity of total joint cost w. r. t. T2 and T4 for n = 3



2. From manufacturer point of view, the total joint cost is $20540 which is $ 120 more than the total integrated cost. If decision is to be taken from raw material's point of view, then the optimal cost is $20470, which is $ 10 more than the total joint cost. For n=3, 4 and 1, the variation in different costs are exhibited in Table 2.

Table 2: Cost variation for n=3, 4 and 1

Cost	(1) n=3	(2) n=4	(3) n=1	(2)-(1)	(3)-(1)
T	0.443	0.463	0.363	0.02	-0.08
Raw material's					
OC	120	120	120	0	0
IHC	15.27	16.73	10.21	1.46	-5.06
PC	2039.42	2135.2	1666.54	95.8	-373
Total=	2174.7	2271.94	1796.7	97.2	-378
TCrm	4908.54	4903.7	4940.56	-4.84	32
Manufacturer's					
OC	90	90	90	0	0
IHC	39.84	43.87	13.24	4.03	-26.6
PC	3081.4	3227.44	2514.12	146	-567
RW	0.013	0.0123	0.0089	-0	-0

Total=	3211.26	3361.3	2617.37	150	-594
TCm	7248.17	7255.11	7197.01	6.94	-51.2
Retailer's					
OC	60	60	60	0	0
IHC	0.28	0.00002	17.045	-0.28	16.8
PC	1031.44	775.2	2793.01	-256	1762
SC	43.62	40.24	70.16	-3.38	26.5
LS	90.85	87.24	115.38	-3.61	24.5
TC/order	1226.19	962.68	3055.6	-264	1829
Total=n*TC/ord	3678.59	3850.77	3055.6	172	-623
TCr	8302.9	8311.5	8401.98	8.6	99.1
GTC=	20460	20470	20540	10	80

3. Increase in number of deliveries increases T1 and T2 and decreases T3 and T4. This because of increasing demand and time dependent deterioration.
4. For n $\geq$ 5, T3 is infeasible meaning that the buyer does not carry any positive stock.
5. For complete backlogging i.e. for B=1, the total joint cost is $ 20742.79.

Sensitivity Analysis:

In table 3, the parameters are varied by -10%, -5%, +5% and 10%. The variations in T2 , T4 , total joint cost are carried out. The last column shows the percentage savings in the total joint cost. The percentage cost savings; PCI is defined as . It is observed that the model is very sensitive to a, Cmr, Cmf, Cr. It shows marginal changes with respect to πrL and B.

Table 3: Sensitivity Analysis

	-10 % changed				-5 % changed			
Parameter	T_2o	T_4^o	TC^o	PCI	T_2^o	T_4^o	TC^o	PCI
Gamma	0.2750	0.1201	20465.5	0.0268	0.2848	0.1202	20462.2	0.0109
a	0.3150	0.1220	18498.6	-9.586	0.3040	0.1211	19479.9	-4.79
b	0.2992	0.1223	20427.2	-0.16	0.2963	0.1213	20443.5	-0.080
c	0.2948	0.1208	20455.6	-0.0214	0.2942	0.1206	20457.65	-0.011
Amo	0.2872	0.1198	20432.3	-0.135	0.2904	0.1201	20446.1	-0.068
Ams	0.2888	0.1199	20439.2	-0.1015	0.2912	0.1201	20449.5	-0.051
Ar	0.2840	0.1195	20418.4	-0.203	0.2888	0.1199	20439.22	-0.1016
hmr	0.2944	0.1204	20456.2	-0.018	0.2940	0.1203	20457.9	-0.0099
hmf	0.2956	0.1205	20450.6	-0.0456	0.2946	0.1204	20455.19	-0.0234
hr	0.2939	0.1201	20459.5	-0.0024	0.2937	0.1202	20459.6	-0.0019
πrB	0.2890	0.1279	20428.1	-0.156	0.2914	0.1240	20444.4	-0.076
πrL	0.2812	0.1272	20395.1	-0.3174	0.2876	0.1237	20428.17	-0.155
Cmrw	0.2936	0.1203	20459.7	-0.001	0.2936	0.1203	20459.7	-0.001
Cmr	0.2963	0.1205	19999.3	-2.252	0.2949	0.1204	20229.5	-1.126

Cmf	0.2993	0.1208	19763.9	-3.4019	0.2964	0.1205	20111.8	-1.701
Crf	0.3153	0.1064	19749.9	-3.4705	0.3051	0.1135	20107.5	-1.72
B	0.1932	0.1635	19986.9	-2.311	0.2560	0.1415	20266.9	-0.9434
μ	0.2936	0.1203	20459.6	-0.001	0.2936	0.1203	20459.6	-0.001
d	0.2936	0.1203	20459.6	-0.001	0.2936	0.1203	20459.6	-0.001
αrm	0.2939	0.1203	20458.6	-0.006	0.2936	0.1203	20459.6	-0.001
αm	0.2946	0.1204	20457.2	-0.0136	0.2941	0.1203	20458.4	-0.007
αr	0.2936	0.1203	20459.7	-0.001	0.2936	0.1203	20459.7	-0.001
βrm	0.2929	0.1202	20463.2	0.0157	0.2933	0.1203	20461.3	0.0065
βm	0.2917	0.1201	20467.46	0.0365	0.2926	0.1202	20463.3	0.0162
βr	0.2934	0.1204	20459.8	-0.001	0.2935	0.1203	20459.7	-0.0014

	-10 % changed				-5 % changed			
Parameter	T_2o	T_4^o	TC^o	PCI	T_2o	T_4^o	TC^o	PCI
γ	0.3014	0.1204	20457.72	-0.011	0.3085	0.1204	20456.18	-0.0186
a	0.2837	0.1195	21437.9	4.779	0.2743	0.1188	22414.6	9.553
b	0.2909	0.1193	20475.72	0.0768	0.2882	0.1184	20491.6	0.154
c	0.2929	0.1201	20461.73	0.0084	0.2923	0.1198	20463.7	0.018
Amo	0.2967	0.1205	20473.1	0.064	0.2997	0.1208	20486.4	0.1295
Ams	0.2959	0.1205	20469.8	0.0479	0.2982	0.1207	20479.8	0.0970
Ar	0.2982	0.1206	20479.8	0.097	0.3027	0.1210	20499.7	0.194
hmr	0.2932	0.1203	20461.42	0.0069	0.2927	0.1202	20463.14	0.0153
hmf	0.2925	0.1202	20464.18	0.0204	0.2915	0.1201	20468.6	0.0423
hr	0.2934	0.1204	20459.79	-0.0009	0.2932	0.1205	20459.8	-0.0006
πrB	0.2956	0.1168	20474.0	0.0684	0.2975	0.1135	20487.4	0.1338
πrL	0.2991	0.1168	20489.7	0.1452	0.3043	0.1133	20518.3	0.2852
Cmrw	0.2936	0.1203	20459.7	-0.001	0.2936	0.1203	20459.7	-0.001
Cmr	0.2922	0.1202	20689.4	1.123	0.2909	0.1201	20919.9	2.248
Cmf	0.2908	0.1201	20807.4	1.697	0.2882	0.1199	21154.9	3.396
Crf	0.2806	0.1267	20805.8	1.690	0.2658	0.1329	21145.16	3.348
B	0.3181	0.1001	20602.6	0.6973	0.3339	0.0807	20710.66	1.225
μ	0.2936	0.1203	20459.6	-0.001	0.2936	0.1203	20459.6	-0.001
d	0.2936	0.1203	20459.6	-0.001	0.2936	0.1203	20459.6	-0.001
αrm	0.2934	0.1203	20460.2	0.001	0.2933	0.1203	20460.7	0.003
αm	0.2931	0.1202	20461	0.005	0.2926	0.1202	20462.2	0.0106
αr	0.2936	0.1203	20459.7	-0.001	0.2936	0.1203	20459.7	-0.001
βrm	0.2938	0.1203	20458.2	-0.0084	0.2941	0.1203	20457.1	-0.014
βm	0.2944	0.1204	20456.5	-0.0168	0.2952	0.1204	20453.8	-0.0302
βr	0.2936	0.1203	20459.7	-0.0015	0.2936	0.1202	20459.6	-0.001

Conclusion

An integrated production-inventory for time varying rate of deterioration of units in inventory system of different players of a supply chain is developed when demand increases quadratically with respect to time. The elapsed time for the production processes shifting to imperfect production is considered to be exponentially distributed. It is observed that the multiple deliveries reduce the total joint cost of an inventory system when compared with an independent decision by the manufacturer or the buyer.

References:

1. Chang K J and Hou K. L (2003), An optimal production-run time with imperfect production process and allowable shortages. *Computers and Operations Research*, 20, 483 – 490.
2. Ghare P M and S F Schrader (1963) A model for exponentially decaying inventory. *Journal of Industrial Engineering*, 14(5), 238-243.
3. Goyal S K and Giri B C (2001), Recent trend in modeling of deteriorating inventory. *European Journal of the Operational Research Society*, 134, 1-16.
4. Goyal S K, Nebebe F (2000) Determination of economic production - shipment policy for a single-vendor-single-buyer system. European *Journal of Operational Research*, 121, 175 - 178.
5. Goyal S K., Eduardo L and Barron C (2002), Note on: Economic production quantity model for items with imperfect quality – a practical approach. *International Journal of Production Economics*, 77, 85 - 87.
6. Hans S, Raafat N I and Paul B L (2006), Joint economic lot-size in distribution system with multiple shipment policy. *International Journal of Production Economics*, 102, 302 - 316.
7. Kim C H and Hong Y (1999), An optimal production run length in deteriorating production processes. *International Journal of Production Economics*, 58, 183 -189.
8. Nahmias S (1982), Perishable inventory theory: A review. *Operations research*, 30, 680-708.
9. Papachristos S and Konstantaras I (2006), Economic ordering quantity models for items with imperfect quality. *International Journal of Production Economics*, 100, 148 - 154.
10. Raafat F (1991), Survey of literature on continuously deteriorating inventory models. *Journal of the Operation Research Society*, 42, 27-37.
11. Rau H, Wu M Y and Wee H M (2003), Integrated inventory model for deteriorating item under a multi echelon supply chain environment. *International Journal of Production Economics*, 86, 155 - 168.
12. Salameh M K and Jaber M Y (2000), Economic production quality model for items with imperfect quality. *International Journal of Production Economics*, 64, 59 - 64.
13. Shah Nita H, Shah Y K (2000), Literature survey on inventory models for deteriorating items. *Economic Annals*. 44 , 221-237
14. Shah Nita H, Gor A S and Wee H M (2008), Optimal joint vendor-buyer inventory strategy for deteriorating items with salvage value. Proceedings of 14th computing: *The Australian theory of symposium* (CATS), 77, 63-68.
15. Woo Y Y, Hsu S L and Wu S (2001), An integrated inventory model for a single vendor and multiple buyers with ordering cost reduction. *International Journal of Production Economics*, 73, 203 - 215.
16. Yang P C and Wee H M (2003), An integrated multi-lot size production inventory model for deteriorating item. *Computers and Operations Research*, 30, 671 - 682.