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# Effect of monitor function in adaptive gid based numerical method for parameterized singular perturbation problem

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***Abstract*--- This works deals with study of a nonlinear singularly perturbed parameterized boundary value problem. The problem is solved by a backward Euler method on an non-uniform mesh. The nonuniform mesh is constructed by using the principle of equidistribution of a monitor function. A comparative is presented for the different choices of the monitor function.**

***Keywords*-- Singularly perturbed parameterized problem, Equidistribution princeple, Adaptive grid, Monitor function, Backward Euler Scheme.**

## I. Introduction

Here, the following singularly perturbed parameterized boundary value problem (SPBVP) is considerd on the domain $\Omega = (0,1)$ :

$$\begin{cases} Tu(x) \equiv \varepsilon y'(x) + f(x,y,\lambda) = 0, \;\; x\epsilon\Omega , \\ \qquad u(0) = a, \;\; u(1) = b \end{cases} \tag{1.1}$$

where$0 < \varepsilon \ll 1$ is the singular perturbation parameter, $\lambda$ is the control parameter $a, b$ are constant. $f(x,y,\lambda)$ is assumed to be sufficiently smooth and satisfies following assumption:

$$\begin{cases} f(x,y,\lambda), \epsilon C^3([0,1] \times \mathbb{R}^2), \\ 0 < \alpha \leq \dfrac{\partial f}{\partial y} \leq \alpha^* < \infty \quad (x,y,\lambda)\epsilon[0,1] \times \mathbb{R}^2 \\ 0 < m \leq \left|\dfrac{\partial f}{\partial \lambda}\right| \leq M < \infty \quad (x,y,\lambda)\epsilon[0,1] \times \mathbb{R}^2 \end{cases} \tag{1.2}$$

The above assumptions ensured the existence of unique solution of the BVP (1.1) (refer [1, 7, 8]). The BVP (1.1) exhibit boundary layer of width $O(\varepsilon)$ near $x = 0$. The parameterized BVP has many application in modeling the various physical phenomena.

In last few years parameterized boundary value problems have been considered by many researchers. A uniformly convergent first order method difference method is developed on a Shishkin mesh for (1.1) in Amiraliyevet. al.[1] and. Cen [2] developed a hybrid difference scheme on Shishkin type meshes and a boundary layer correction technique is used to solve the problem of the form (1.1) in Xie et. al. [10].

In this article, we study the effect of choice of monitor function in implementing the difference scheme on a nonuniform mesh known as adaptive grid [4, 6]. A comparative study is presented that how the choice of monitor function can affect in reducing the maximum pointwise error. Throughoutthis paper C denote a generic positive constant.

## II. Analytical properties

**Lemma 2.1** *The solution {y(x), λ} of (1.1) satisfies the following the inequalities*:

$$|\lambda| \leq C, \;\; |u^k(x)| \leq C\left\{1 + \varepsilon^{-k} \, exp\left(-\frac{\alpha x}{\varepsilon}\right)\right\}, x\epsilon\bar{\Omega}\,, \quad k = 0,1\;2,3$$

*Proof.* See the proof in [1,2].

## III. Discretization and MEsh

### A. *Discrete problem*

Consider difference approximations for BVP (1.1) on a non-uniform partition $\Omega^N = \{0 = x_0 < x_1 < \cdots x_{N-1} < x_N = 1\}$, and denote $h_i = x_i - x_{i-1}$; $i = 1, 2, \cdots, N$., we Now, the Backward Euler scheme for(1.1) takes theform,

$$\begin{cases} L^N Y_j \equiv \varepsilon D^- Y_j + f(x_j, Y_j, \lambda^n) = 0, \quad 1 \le j \le N-1, \\ Y_0 = a, \quad Y_N = b. \end{cases} \qquad (3.1)$$

Here, $D^-\phi_j = \frac{\phi_i - \phi_{i-1}}{h_i}$ for any mesh function $\phi_i$.

*B. Mesh generation*

To get uniformly convergent numerical approximation, one has to use layer adapted nonuniform mesh, which are fine inside the boundary layer region and coarse in the outerregion. Such a grid can be generated by using equidistribution of a positive monitor function. A grid $\Omega^N$ satisfies following:

$$\int_{x_{i-1}}^{x_i} M(y(s), s)ds = \int_{x_i}^{x_{i+1}} M(y(s), s)ds, \quad i = 1, \dots, N-1, \qquad (3.2)$$

is said to be equidistribution, where$M(y(s), s) > 0$ is called a monitor function. Here, we study the following two kind of monitor function to construct a nonuniform mesh

(i) $M(y(x), x) = \sqrt{1 + (y'(x))^2}$,

(ii) $M(y(x), x) = 1 + |y''(x)|^{\frac{1}{2}}$.

The following well discussed adaptive algorthim is used to construct the nonuniform mesh(refer [4,8,9]).

## IV. *ADAPTIVE MESH GENERATION ALGORITHM*

**Step 1:** Take the initial mesh $\{x_i^0 = \{0, \frac{1}{N}, \frac{2}{N}, \dots, 1\}$ as uniform mesh.

**Step 2:** Compute the discrete solution $y_i^{N,(k)}$ on mesh $\{x_i^k\}$, for $= 0,1,2, \dots$ .

**Step 3:** Find the discretized monitor function $M_i^{(k)}$ and compute $L_j^{(k)} = \sum_{i=1}^{j} h_i^{(k)} M_i^{(k)}$.

**Step 4:** Let $C_0$ be the user chosen constant, where $C_0 > 1$. If $\frac{max_{i=1,\dots,N} h_i^{(k)} M_i^{(k)}}{L_N^{(k)}} \le \frac{C_0}{N}$, then go to Step 6, otherwise continue to Step 5.

**Step 5:** Generate a new mesh by equidistributing the proposed monitor function . Set $Y_i^{N,(k)} = \frac{iL_N^{(k)}}{N}, for\ i = 0,1, \dots, N.$ Now interpolate $\left(x_i^{(k+1)}, Y(x_i^k)\right)$ to $(x_i^k, M_i^k)$ using piecewise linear interpolation. Generate a new mesh $\{x_i^{(k)} = \{0 = x_0^{(k+1)}, x_1^{(k+1)}, \dots, x_N^{(k+1)} = 1\}$ and return to Step 2.

**Step 6:** Set $\{x_i^{(k)} = \{0 = x_0^{(k)}, x_1^{(k)}, \dots, x_N^{(k)} = 1\}$ and $Y = Y^k$ . **Stop.**

## V. MAIN RESULT

**Theorem 4.1** *Let* $\{y(x), \lambda\}$ *and* $\{Y_j^N, \ \lambda^N\}$ *be the exact solution and discrete solution respectively. Then,*

$$\underbrace{max}_{j} |y(x_j) - Y_j^N| < CN^{-1}, \quad |\lambda - \lambda^N| < CN^{-1}, \qquad (4.1)$$

*where C is a constant independent of $N$ and $\varepsilon$*

## VI. NUMERICAL EXAPERIMENT AND DISCUSSION

The following test problem is taken for the numerical discussion.

**Example 5.1** *Consider the following nonlinear singularly perturbed problem*

$$\begin{cases} \varepsilon y'(x) + 2y - \exp(-y) + x^2 + \lambda + \tanh(\lambda + x) = 0, \\ x \epsilon \Omega = (0,1), \\ \quad y(0) = 1, \ y(1) = 0. \end{cases} \qquad (5.1)$$

The exact solution is not available for the parameterized BVP (5.1). Thus, to calculate the maximum pointwise error $E_{\varepsilon,u}^N$ and the rate of convergence $r_{\varepsilon,u}^N$, we use the double mesh principle. Let us define $\bar{Y}_i^{2N}$ as piecewise linear interpolation to $Y_i^N$ in $\Omega^N$. For any value of N, maximum pointwise error with respect to the variable u is defined as $E_{\varepsilon,u}^N = \underbrace{max}_{j} |Y_j^N - \bar{Y}_j^{2N}|$. Similarly for the parameter $\lambda$, the maximum pointwise error is defined as $E_{\varepsilon,\lambda}^N = \underbrace{max}_{j} |\lambda^N - \bar{\lambda}^{2N}|$. The corresponding rate of convergenceare calculated by

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$$r_{\varepsilon,u}^{N} = log_2\left(\frac{E_{\varepsilon,u}^{N}}{E_{\varepsilon,u}^{2N}}\right), \quad r_{\varepsilon,\lambda}^{N} = log_2\left(\frac{E_{\varepsilon,\lambda}^{N}}{E_{\varepsilon,\lambda}^{2N}}\right).$$

The results are presented in Table 1 and in Table 2 which are clear illustrations of the convergence estimate. Morover it can be observed
that the maximum pointwise error is less while using $1 + y''(x)|^{\frac{1}{2}}$ as monitor function instead of $\sqrt{1+(y'(x))^2}$. However the approximation converges with linear rate of convergence for both the monitor functions but the error is less than when the monitor function involves second order derivative.

Table I: For $M(y(x), x) = \sqrt{1+(y'(x))^2}$.

| N | $\varepsilon = 1e-4$ | $\varepsilon = 1e-8$ |
|---|---|---|
| 16 | 0.013692<br>0.67 | 0.013705<br>0.68 |
| 32 | 0.0085672<br>0.85 | 0.0085717<br>0.85 |
| 64 | 0.004763<br>0.91 | 0.0047648<br>0.91 |
| 128 | 0.0025373<br>0.93 | 0.0025386<br>0.92 |
| 256 | 0.0013363<br>0.95 | 0.0013374<br>0.95 |
| 512 | 0.00069033<br>0.97 | 0.000691010.97 |
| 1024 | 0.00035261 | 0.00035325 |

Table II: For $M(y(x), x) = 1 + |y''(x)|^{\frac{1}{2}}$.

| N | $\varepsilon = 1e-4$ | $\varepsilon = 1e-8$ |
|---|---|---|
| 16 | 0.0089624<br>0.56 | 0.0089615<br>0.56 |
| 32 | 0.0060763<br>0.66 | 0.0060757<br>0.66 |
| 64 | 0.0038574<br>0.72 | 0.0038571<br>0.72 |
| 128 | 0.0023363<br>0.77 | 0.0023361<br>0.77 |
| 256 | 0.0013663<br>1.0079 | 0.0013662<br>1.0079 |
| 512 | 0.0006794<br>1.0137 | 0.00067934<br>1.0136 |
| 1024 | 0.00033648 | 0.00033646 |

## VII. CONCLUDING REMARKS

In this articel, effect of choice of monitor function is studied for the implementation of adaptive grid based backward Euler difference scheme for solving parameterized SPBVP. It has been observed that the choice of monitor function also improve the accuracy of the approximation.

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